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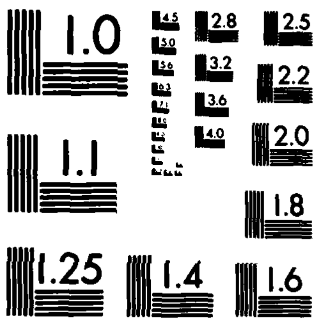

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Optimal Designs For Comparisons Between Two Sets Of Treatments

By

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Optimal Designs For Comparisons Between Two Sets of Treatments

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Abstract

Suppose v treatments are to be compared in b blocks of size k each. Also suppose that the treatments are divided into 2 sets of u and w = v - u treatments. A-optimal designs are obtained for estimating all the differences of two treatments, one from each set. Optimal row-column designs are also obtained. Some new optimal designs for comparing several treatments with a single control are obtained as special cases.

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1. Introduction

We consider an experiment to compare v treatments in b blocks of size k each. The model of observations is assumed to be linear, additive and homoscedastic. The problem is to determine an optimal design, that is, an allocation of treatments to blocks that is best in some sense.

The criterion of optimality depends on the objective of the experiment. If the objective is to compare all treatments with one another, then Kiefer, in a series of well known publications, and other authors have determined families of optimal designs. One celebrated result is that a BIB design is optimal in any reasonable sense (Kiefer (1975)).

In many experimental situations, all comparisons between treatments are not equally important. For example, one treatment may be a standard treatment or a control, enjoying a special status. The rest of the treatments (test treatments) are to be compared with the control - comparisons among test treatments not being of much importance. Some optimal designs for these experiments are now available in the literature. To get started in the area, the reader may look at Pearce (1960), Bechhoffer and Tamhane (1981), Majumdar and Notz (1983), Giovagnoli and Wynn (1983), Constantine (1983), Hedayat and Majumdar (1984), Jacroux (1984), Notz (1984). This is a list of some, though not all, important papers.

More generally, the treatments may be divided into two sets, say G and H containing u and $w = v - u$ treatments respectively. All comparisons of two treatments, one from G and one from H are important, but comparisons within G or within H are of less consequence. In case one treatment is a control, we take $u = 1$. Similarly, experiments could be conducted to compare several treatments with two or more standard treatments. For example, a factory may be using two types of equipments for a job. The object of an experiment could be to determine whether some or all of three types of new equipments are better than the existing ones. In this situation of several controls, both u and w are 2 or more. Farmers often use several varieties at the same time, since each of them could be suseptible to a different disease. Thus, in agricultural experiments, the object could be to determine which "package" of varieties perform better; or whether some variety from one package should be replaced by a variety from another.

In this article we consider the problem of finding optimal designs for these and similar situations. Cox (1971, p. 238) suggests that if there are several controls, then each of them may be used once in each block. The rest of the design would be a BIB design in the treatments from the other set. We shall formulate the problem mathematically in the tradition of Kiefer. Our criterion will be the sum of the variances of the treatment differences, one from each set. This criterion is statistically meaningful in this context. We shall derive several methods of obtaining optimal designs - applicable under different types of experimental situations.

In Section 2 we look at experimental situations with blocks of small size; that is, k is small compared to u and w. We develop a method of obtaining optimal designs in Theorem 2.1 and obtain some additional refinements for the case $u = w$, in Corollary 2.2. When $u = w$, intuition suggests that the best design might have half of each block filled by treatments from G, and the other half from H (some adjustments being needed for odd k's). While we can essentially prove this for $k \geq 4$, surprisingly it is not true when $k = 2$. We also give a catalog of optimal designs for some values of u and w when $k = 2$ and $b \leq 100$.

In Section 3, we look at blocks of large size. The method of proving optimality is entirely different here. We prove a theorem applicable to any linear model, and derive optimal block designs in a corollary. This also yields previously unknown results for the case of a single control.

The general theorem of Section 3 is applied to models eliminating two sources of heterogeneity in Section 4. This gives us optimal row - column designs. As a consequence we could generalize some results of Notz (1984) for the case of a single control.

2. Optimal Designs in Small Blocks.

We consider the problem of comparing v treatments in b blocks of size k each. The treatments can be divided into two sets G and H of u and w $(=v-u)$ treatments respectively. For this section only, we assume that k is small; indeed we insist that

$$k \leq \min\,(u, w) \tag{2.1}$$

The model is additive, homoscedastic and linear. This means that if $Y_{ij\ell}$ is the observation on treatment i $(1 \leq i \leq v)$ in block j $(1 \leq j \leq b)$ and plot ℓ $(1 \leq \ell \leq k)$, then

$$Y_{ij\ell} = \tau_i + \beta_j + \epsilon_{ij\ell}, \tag{2.2}$$

where $\epsilon_{ij\ell}$ are assumed to be uncorrelated random variables with mean 0 and common variance σ^2. The unknown constants τ_i and β_j represent the effect of treatment i and the effect of block j respectively. Let $\mathcal{D}(u, w, b, k)$ be the set of all possible experimental designs. The purpose of the experiment is to compare each treatment from G with each treatment from H. Comparisons within G and H are of secondary importance and will play no role in the selection of a design. We want to choose an experimental design from $\mathcal{D}(u, w, b, k)$ which minimizes

$$\sigma^{-2} \sum_{g\in G} \sum_{h\in H} \mathrm{Var}(\hat{\tau}_{dg} - \hat{\tau}_{dh}) \tag{2.3}$$

as d varies over all of $\mathcal{D}(u, w, b, k)$. Here $\hat{\tau}_{dg} - \hat{\tau}_{dh}$ denotes the BLUE of $\tau_g - \tau_h$ under design d. This criterion can be called the A-criterion since the optimal design minimizes the average variance of the contrasts of interest, $\tau_g - \tau_h$, $g \in G$, $h \in H$. The A-criterion is expected to be generally acceptable in most situations since it is statistically meaningful.

If $d \in \mathcal{D}(u, w, b, k)$, then we denote by $n_{dij} (0 \le n_{dij} \le k)$ the number of times treatment i occurs in block j. $N_d = (n_{dij})$ is the incidence matrix of d and $C_d = D_{dr} - (1/k)N_d N_d'$ is the "C-matrix." Here D_{dr} is a diagonal matrix consisting of the replications of the treatments.

Let us denote by 1_n and I_n the vector of n ones and the $n \times n$ identity matrix respectively, and let $J_{mn} = 1_m 1_n'$. Write

$$P_i = [0 \cdots\cdots 1_w \cdots\cdots 0 \; \vdots \; -I_w] ,$$

a $w \times v$ matrix in partitioned form. If

$$P' = [P_1' \ \ldots \ P_u']$$

then (2.3) can be written as a function of C_d in the form

$$\phi(C_d) = \mathrm{tr}\ P\, C_d^- P' = \sum_{i=1}^{u} \mathrm{tr}\ \ P_i C_d^- P_i',$$

where C_d^- is any generalized inverse of C_d. The following is a useful subclass of $v \times v$ matrices.

$\mathcal{C} = \{C : C$ is nonnegative definite, rank $C = v - 1$, $C\, 1_v = 0$ and for some scalars p, q, r, s and t, C can be written as

$$C = \begin{pmatrix} p\, I_u + q\, J_{uu} & t\, J_{uw} \\ t\, J_{wu} & rI_w + J_{ww} \end{pmatrix},$$

For each $d \in \mathcal{D}(u, w, b, k)$, we define

$T_{d1j} = \sum_{g \in G} n_{dgj}$, $T_{d2j} = \sum_{h \in H} n_{dhj}$, $T_{d1} = \sum_{j=1}^{b} T_{d1j}$,

$T_{d2} = \sum_{j=1}^{b} T_{d2j}$, $S_{d1} = \sum_{j=1}^{b} T^2_{d1j}$ $S_{d2} = \sum_{j=1}^{b} T^2_{d2j}$. We are now in a position to state and prove the main theorem of this section.

<u>Theorem 2.1</u>. Suppose $d_0 \in \mathcal{D}(u, w, b, k)$ is such that (i) $C_{d_0} \in \mathcal{C}$, (ii) $n_{d_0 ij} = 0$ or 1 for all i,j and (iii) $f(T_{d_0 1}, S_{d_0 1}) = \min_{d \in \mathcal{D}(u, w, b, k)} f(T_{d1}, S_{d1})$ where $f(T_{d1}, S_{d1}) = 1/(kT_{d1} - S_{d1}) + (w-1)^2/[(w(k-1)-k)T_{d2} + S_{d2}] + (u-1)^2/[(u(k-1)-k)T_{d1} + S_{d1}]$, then d_0 is A-optimal in $\mathcal{D}(u, w, b, k)$.

<u>Proof</u> Let Π be the set of all permutations π of the treatments which can be expressed as a product $\pi = \pi_g \pi_h$, where π_g is a permutation of G and π_h a permutation of H. Let $Q(\pi)$ be the corresponding permutation matrix. Then it is easy to see that

$$\phi(C_d) = \phi(Q(\pi) C_d\, Q(\pi)'),$$

Moreover, $\phi(C_d)$ is convex in C_d. These two properties imply that

$$\phi(C_d) \quad \geq \quad \phi(C_d^*)$$

where $C_d^* = [\Sigma\phi(Q(\pi)C_dQ(\pi)')]/u!w!$ where Σ is over all π in Π.

Clearly $C_d^* \in C$ with $p + q = \sum_{g=1}^{u} c_{dgg}/u$, $q = \sum_{g \neq g'=1}^{u} c_{dgg'}/u(u-1)$, $r + s = \sum_{h=u+1}^{v} c_{dhh}/w$, $s = \sum_{h \neq h'=u+1}^{v} c_{dhh'}/w(w-1)$ and $t = \sum_{g=1}^{u}\sum_{h=u+1}^{v} c_{dgh}/uw$. Here, we write $G = \{1,2,\ldots,u\}$, $H = \{u+1,\ldots,v\}$. To evaluate $\phi(C_d^*)$ we must find the eigenvalues and eigenvectors of C_d^*. The eigenvalues can be obtained from Constantine (1981). In fact, we can express C_d^* in its spectral form as

$$C_d^* = pE_1 + rE_2 - (vt)E_3,$$

where $E_1 = \begin{pmatrix} I_u - u^{-1} J_{uu} & 0 \\ 0 & 0 \end{pmatrix}$,

$$E_2 = \begin{pmatrix} 0 & 0 \\ 0 & I_w - w^{-1}J_{ww} \end{pmatrix}, \quad uwvE_3 = \begin{pmatrix} w^2 J_{uu} & -uwJ_{uw} \\ -uwJ_{wu} & u^2 J_{ww} \end{pmatrix}$$

The Moore-Penrose inverse of C_d^* is

$$C_d^{*+} = p^{-1} E_1 + r^{-1}E_2 - (vt)^{-1}E_3.$$

Hence,

$\phi(C_d^*) = \text{trace } PC_d^{*+}P' = w(u-1)/p + u(w-1)/r - 1/t$, with

$$p = T_{d1}/u - \sum_{j=1}^{b}\sum_{g=1}^{u} (n_{dgj} - T_{d1j}/u)^2/k(u-1),$$

$$r = T_{d2}/w - \sum_{j=1}^{b} \sum_{h=u+1}^{v} (n_{dhj} - T_{d2j}/w)^2/k(w-1) ,$$

$$t = - \sum_{j=1}^{b} T_{d1j} T_{d2j}/uwk = -(kT_{d1}-S_{d1})/uwk.$$

Let us fix $T_{d11},\ldots,T_{d1b}$ and try to reduce $\phi(C_d^*)$. Since, by condition (2.1), $T_{d1j}/u \le k/u \le 1$,

$$\sum_{j=1}^{b} \sum_{g=1}^{u} (n_{dgj} - T_{d1j}/u)^2 \ge T_{d1} - S_{d1}/u,$$

with equality whenever $n_{dgj} = 0$ or 1, for all g and j. Similarly, since $T_{d2j}/w \le 1$,

$$\sum_{j=1}^{b} \sum_{h=u+1}^{v} (n_{dhj} - T_{d2j}/w)^2 \ge T_{d2} - S_{d2}/w,$$

with equality whenever $n_{dhj} = 0$ or 1, for all h and j. These relations yield a lower bound to $\phi(C_d^*)$ for fixed values of $T_{d11},\ldots,T_{d1\ell}$. After some simplifications, we get

$$\phi(C_d) \ge \phi(C_d^*) \ge u\, w\, f(T_{d1},S_{d1}).$$

Hence the theorem.

Remark 1. We can write $f(T_{d1},S_{d1})$ as a function of T_{d1} and S_{d1} only since $T_{d1} + T_{d2} = bk$ and $kT_{d1} - S_{d1} = k\,T_{d2} - S_{d2}$.

To carry out the minimization of $f(T_{d1},S_{d1})$ it could be useful to introduce integers $b_{d\alpha}(\alpha=0,1,\ldots,k)$, where

$b_{d\alpha}$ is the number of blocks for which $T_{d1j} = \alpha$. In this notation $b = \sum_{\alpha=0}^{k} b_{d\alpha}$, $T_{d1} = \sum_{\alpha=0}^{k} \alpha b_{d\alpha}$ and $S_{d1} = \sum_{\alpha=0}^{k} \alpha^2 b_{d\alpha}$. The problem reduces to minimizing $f(T_{d1},S_{d1}) = f^*(b_{d0},\ldots,b_{dk})$ over integers $b_{d\alpha}$ $(\alpha = 0,1,\ldots,k)$, where each $b_{d\alpha} = 0,1,\ldots,b$ and $\sum_{\alpha=0}^{k} b_{d\alpha} = b$. If k is small then this problem is not too time consuming. For example, when $k = 3$, $u = 3$, $w = 4$ and $b = 30$, the optimal values b_α of $b_{d\alpha}$ are $b_0 = 0$, $b_1 = 18$, $b_2 = 12$ and $b_3 = 0$. For a clearer representation, we shall denote the treatments in G by numbers $(1,2,\ldots)$ and the treatments in H by alphabets $(A,B,\ldots)$. The following design, with columns as blocks, is A-optimal in $\mathcal{D}(3,4,30,3)$.

1	1	2	1	1	2	1	1	2	1	1	1	1	1	1	2	2	2	2	2	2	3	3	3	3	3	3
2	3	3	2	3	3	2	3	3	A	A	A	B	B	C	A	A	A	B	B	C	A	A	A	B	B	C
A	A	A	B	B	B	C	C	C	B	C	D	C	D	D	B	C	D	C	D	D	B	C	D	C	D	D

One attractive feature of this example is that only two optimal b_α's are positive. A study of $f(T_{d1},S_{d1})$, with some examples, shows that this may not be true, in general. Thus, there is no guarantee that the treatments from H, as a collection, are spread as homogeneously as possible over the blocks, which often happens when $u = 1$, the single control case (Majumdar and Notz (1984)).

When $k = 2$, the problem is to minimize $1/b_{d1} + (u-1)/(b_{d1} + (2u/(u-1))b_{d2}) + (w-1)/(b_{d1} + (2w/(w-1))b_{d0})$, over integers in the region $b_{d0} + b_{d1} + b_{d2} = b$. If the optimal $b_{d\alpha}$ is denoted by b_α, then the necessary and sufficient conditions for obtaining an optimal design by the method of Theorem 2.1 are $b_0 \equiv 0 (\mathrm{mod}\ w(w-1)/2)$, $b_1 \equiv 0\ (\mathrm{mod}\ uw)$ and $b_2 \equiv 0\ (\mathrm{mod}\ u(u-1)/2)$. We shall denote by $a_1 \Sigma a_2$ the set of all a_2-ples which can be chosen from a set of a_1 treatments. Also let d_1 denote m_1 copies of $w \Sigma 2$ in the treatments from H, d_3 denote m_3 copies of $u \Sigma 2$ from G and d_2 denote m_2 copies of the uw pairs of treatments formed by matching a treatment from G with a treatment from H. Here $b_0 = m_1\ w(w-1)/2$, $b_1 = m_2\ uw$ and $b_2 = m_3\ u(u-1)/2$. When the necessary and sufficient conditions are satisfied the A-optimal design is

$$d = d_0 \cup d_1 \cup d_2.$$

We searched and found A-optimal designs for $k = 2$ in the range $v \le \ell \le 100$ for treatments $(u,w) = (2,2), (2,3), (2,4), (2,5), (2,6), (3,3), (3,4)$. The catalog of A-optimal designs produced by Theorem 2.1 for these values of the parameters are given in Table 1. This table gives the optimal values b_0, b_1 and b_2 for combinations of u, w and b. For example, when $u = 2$, $w = 3$, and $b = 22$, Table 1 shows that $b_0 = 3$, $b_1 = 18$ and $b_2 = 1$. Therefore an A-optimal design in $\mathcal{D}(2,3,22,2)$ is

1 1 1 1 2 2 2 1 1 1 2 2 2 1 1 1 2 2 2 A A B
2 A A A A A A B B B B B B C C C C C C B C C

Actually, our search was more extensive. It included all (u,w, b) values with $2 \leq u \leq 6$, $u \leq w \leq 6$ and $v \leq b \leq 100$. For some of these (u,w) combinations no optimal design could be obtained by Theorem 2.1 for $b \leq 100$.

Given u,w and b, Theorem 2.1 could give an optimal design. In case it fails to do so, it still gives a lower bound to the A-value, $\phi(C_d)$, for a $d \in \mathcal{L}(u,w,b)$. Usually the experimenter has a fairly good idea of what an efficient design looks like. By comparing the A-value of a design with the lower bound given by Theorem 2.1 he will be able to make a good assessment of the efficiency of his design. At the very least, this is expected to be the practical usefulness of Theorem 2.1.

The combinatiorial restrictions imposed on d_0 by the condition $C_{d_0} \in C$ are quite severe. Construction of an A-optimal design will, in general, be a difficult task.

In case $w = u$, one can obtain a simplification of Theorem 2.1. This is given in Corollary 2.2.

Corollary 2.2. Suppose $w = u$ and $k \geq 4$. Let $d_0 \in \mathcal{L}(u,u,b,k)$ be such that

(i) $n_{d_0ij} = 0$ or 1, $i = 1,\ldots,v$, $j = 1,\ldots,b$.

(ii) $C_{d_0} = \begin{pmatrix} pI_u + qJ_{uu} & t\,J_{uu} \\ t\,J_{uu} & pI_u + qJ_{uu} \end{pmatrix}$

for some p, q and t.

(iii) $T_{d_0 1j} = k/2$ when k is even

$= [k/2]$ or $[k/2] + 1$ when k is odd

for each $j = 1,\ldots,b$, where $[m]$ denotes the largest integer not exceeding m. Then d_0 is A-optimal in $\mathcal{D}(u,u,b,k)$.

Proof. Let $d \in \mathcal{D}(u,u,b,k)$. Start by defining C_d^*, the average of C_d over all permutations in Π, as in the proof of Theorem 2.1. Then define

$$C_{d1}^* = \begin{pmatrix} 0 & I_u \\ I_u & 0 \end{pmatrix} C_d^* \begin{pmatrix} 0 & I_u \\ I_u & 0 \end{pmatrix}$$

and $C_d^{**} = (C_d^* + C_{d1}^*)/2$. Observe that $\phi(C_{d1}^*) = \phi(C_d^*)$. By the convexity of ϕ,

$$\phi(C_d) \geq \phi(C_d^*) \geq \phi(C_d^{**}).$$

Clearly, $C_d^{**} = \begin{pmatrix} pI_u + qJ_{uu} & t\,J_{uu} \\ t\,J_{uu} & pI_u + qJ_{uu} \end{pmatrix}$, where $p + q = \sum_{i=1}^{v} c_{dii}/v$,

$$q = \Big(\sum_{g \neq g'=1}^{u} \sum c_{dgg'} + \sum_{h \neq h'=u+1}^{v} \sum c_{dhh'} \Big) / 2u(u-1),$$

$$t = \Big(\sum_{g=1}^{u} \sum_{h=u+1}^{v} c_{dgh} \Big) / u^2.$$

$$\phi(C_d^{**}) = 2u(u-1)/p - 1/t.$$

Some computations and an appeal to condition (2.1) yield,

$$p \leq (b\,k(k(u-1) - u) + S_d)/ku(u-1)$$

with equality whenever $n_{dij} = 0$ or 1 for all i, j. Here $S_d = S_{d1} + S_{d2}$. Recall that $t = -\sum_{j=1}^{b} T_{d1j} T_{d2j}/ku^2 = -\gamma_d/ku^2$ where we write $\gamma_d = \sum_{j=1}^{b} T_{d1j} T_{d2j}$. Since $\gamma_d = (bk^2 - S_d)/2$,

$$\phi(C_d^{**}) \geq ku^2 \{1/\gamma_d + 4(u-1)^2/(buk(k-1) - 2\gamma_d)\}.$$

It is easy to see that, Max γ_d is achieved when

$$T_{d1j} = k/2 \qquad \text{when } k \text{ is even}$$

$$= [k/2] \text{ or } [k/2] + 1 \qquad \text{when } k \text{ is odd},$$

for each $j = 1,\ldots,b$. In either case, Max $\gamma_d \leq bk^2/4$. Let us study the function

$$a(\gamma_d) = 1/\gamma_d + 4(u-1)^2/(buk(k-1) - 2\gamma_d).$$

$a(\gamma_d)$ is convex in γ_d and

$$\partial a(\gamma_d)/\partial \gamma_d = a_1(\gamma_d)/(\gamma_d^2 (buk(k-1) - 2\gamma_d)^2)$$

where $a_1(\gamma_d) = 8(u-1)\gamma_d^2 - (buk(k-1) - 2\gamma_d)^2$. Using some calculus one can show that, for each $u \geq 2$, $a_1(bk^2/4) < 0$ whenever $k \geq 4$. Hence $a(\gamma_d)$ is a minimum when γ_d is a maximum. This establishes Corollary 2.2.

Table 1 shows that Corollary 2.2 need not hold when $k = 2$. For $k = 3$, the question is open.

In some situations an experimenter may be interested only in comparing the sets of treatments G and H each taken as a whole. In agricultural experiments the object could be to determine which of the groups, G or H, results in the best overall yield. Here the problem could be to compare the averages $\sum_{g\in G} \tau_g/u$ and $\sum_{h\in H} \tau_h/w$. So, we have to choose a d in $\mathcal{D}(u,w,b,k)$ which minimizes $\mathrm{Var}(\sum_{g\in G} \hat{\tau}_{dg}/u - \sum_{h\in H} \hat{\tau}_{dh}/w)$. Along the lines of Theorem 2.1 it can be shown that if $C_{d_0} \in \mathcal{C}$ and $T_{d_0 1j} = k/2$ if k is even and $T_{d_0 1j} = [k/2]$ or $[k/2] + 1$ if k is odd for all j, then d_0 is optimal. In fact, no restriction on k, like condition (2.1), is necessary for this result.

3. Optimal Designs In Large Blocks

In this section k will usually be large, so that condition (2.1) is not assumed any more. We shall start by proving a general result for Gauss Markov models, and then derive A-optimal designs for particular models.

Let $\mathcal{D}$ be the set of all designs. If $d \in \mathcal{D}$ then the observations Y_d will have the Gauss Markov model

$$E(Y_d) = X_{1d}\theta_1 + X_{2d}\theta_2, \quad Var(Y_d) = \sigma^2 I.$$

Here the vectors θ_1, θ_2 and the scalar σ^2 are the unknown parameters. We will assume that $X'_{1d}X_{1d}$ is nonsingular for all $d \in \mathcal{D}$. Define

$$C_d = X'_{1d}X_{1d} - X'_{1d}X_{2d}(X'_{2d}X_{2d})^{-}X'_{2d}X_{1d}$$

If ψ is a real valued function defined on nonnegative definite matrices then d_0 will be said to be ψ - optimal for estimating $Q\theta_1$, if

$$\psi(QC^{-}_{d_0}Q') \le \psi(Q\ C^{-}_{d}Q'), \quad \text{for all} \quad d \in \mathcal{D}.$$

<u>Theorem 3.1</u>. Let ψ have the property that $\psi(A-B) \le \psi(A)$ whenever A, B and $A-B$ are nonnegative definite. Suppose $d_0 \in \mathcal{D}$ satisfies

$$X'_{2d_0}X_{1d_0}(X'_{1d_0}X_{1d_0})^{-1}Q' = 0 \tag{3.1}$$

and

$$\psi(Q(X'_{1d_0}X_{1d_0})^{-1}Q') \le \psi(Q(X'_{1d}X_{1d})^{-1}Q') \tag{3.2}$$

$$\text{for all} \quad d \in \mathcal{D}.$$

Then d_0 is ψ-optimal for estimating $Q\theta_1$.

<u>Proof</u>. Suppose A is a positive definite matrix, B is a nonnegative definite matrix and $C = A - B$ is also nonnegative definite. Let Q be a matrix with $\mathfrak{m}(Q') \subset \mathfrak{m}(C)$, where $\mathfrak{m}$ denotes the column span of a matrix. Then it can be shown that

$$QC^{-}Q' = QA^{-1}Q' \quad \text{if} \quad BA^{-1}Q' = 0.$$

Hence for every $d \in \mathcal{D}$,

$$\begin{aligned} \phi(QC_{d_0}^{-}Q') &= \phi(Q(X_{1d_0}'X_{1d_0})^{-1}Q') \quad \text{by (3.1)}, \\ &\le \phi(Q(X'_{1d}X_{1d})^{-1}Q') \quad \text{by (3.2)}, \\ &\le \phi(QC_d^{-}Q'), \end{aligned}$$

since there exists a generalized inverse C_d^{-} for which $C_d^{-} - (X_{1d}'X_{1d})^{-1}$ is nonnegative definite. This establishes the theorem.

We shall apply this theorem to the block design model given in (2.2). Here $\theta_1' = (\tau_1,\ldots,\tau_v)$ and $\theta_2' = (\beta_1,\ldots,\beta_b)$. For a design $d \in \mathcal{D}(u,w,b,k)$, $X'_{1d}X_{1d} = D_{dr}$, the diagonal matrix showing the replications $r_{d1},\ldots,r_{dv}$ of the treatments; $X_{2d}'X_{2d} = kI_b$; $X_{1d}'X_{2d} = N_d$, the treatment block incidence matrix. Let the treatments be divided into two sets G and H, as in Section 2, and suppose Q is the matrix P defined in Section 2, ϕ = trace. Equation (3.1) is satisfied whenever

$$N_d' D_{dr}^{-1} = \Delta J_{bv}, \tag{3.3}$$

for some diagonal matrix Δ. To find a design d_0 which satisfies (3.2) one has to minimize

$$\sum_{g=1}^{u} \sum_{h=u+1}^{v} \left(\frac{1}{r_{dg}} + \frac{1}{r_{dh}}\right) \tag{3.4}$$

over all designs $d \in \mathcal{D}(u,w,b,k)$. In (3.4) we are actually minimizing $\operatorname{tr} V(P\hat{\theta}_{1d})$, but for a model with no block effects $\beta_1,\ldots,\beta_b$, while (3.3) is similar to a treatment - block orthogonality condition. Corollary 3.2 gives one class of optimal designs.

Corollary 3.2. Suppose $k \equiv 0(\text{mod } (u + \sqrt{uw}))$ and $k \equiv 0(\text{mod } (w + \sqrt{uw}))$. Let $d_0 \in \mathcal{D}(u,w,b,k)$ satisfy

$$n_{d_0 gj} = k/(u + \sqrt{uw}), \quad n_{d_0 hj} = k/(w + \sqrt{uw}), \tag{3.5}$$

for $g = 1,\ldots,u$, $h = u + 1,\ldots, v$, $j = 1,w,\ldots,b$. Then d_0 is A-optimal in $\mathcal{D}(u,w,b,k)$.

Proof. Consists of verification of (3.3) and a straightforward solution of (3.4) using Lagrange's multipliers.

Remark. In particular when $u = 1$, we have the case of a single control in G and $w = v - 1$ test treatments in H. Corollary 3.2 says that as long as

$$n_{d_0 ij} = \sqrt{w}\, n_{d_0 1j},$$

for each $i = 2,\ldots,v$ and $j = 1,\ldots,b$, then the resulting design is A-optimal for control-test treatment comparisons. Here the control is denoted by 1, and the test treatments by $2,\ldots,v$. To our knowledge, the result is not available in the literature.

4. Optimal Row Column Designs.

Suppose v treatments are to be compared in an $r \times c$ array. The response Y_{ijt} in row j and column t is

$$Y_{ijt} = \rho_j + \eta_t + \tau_i + \epsilon_{ijt},$$

if treatment i is used in cell (j,t); $i = 1,\ldots,v$, $j = 1,\ldots,r$, $t = 1,\ldots,c$. Here ρ_j is a row effect, η_t is a column effect, τ_i is a treatment effect and ϵ_{ijt} is a random error. As usual, ϵ's are assumed to be uncorrelated, with zero expectation and variance σ^2. The treatments are divided into two sets G and H of u and w treatments respectively. $\mathcal{D}(u,w,r,c)$ denotes the set of all designs.

We shall apply Theorem 3.1 to this setup. Take $\theta_1' = (\tau_1,\ldots,\tau_v)$ and $\theta_2' = (\rho_1,\ldots,\rho_r, \eta_1,\ldots,\eta_c)$. As in Section 3, $d_0 \in \mathcal{D}(u,w,r,c)$ is A-optimal if

$$N'_{1d_0} D^{-1}_{d_0} r = \Delta_1 J_{rv}$$
$$N'_{2d_0} D^{-1}_{d_0} r = \Delta_2 J_{cv}, \tag{4.2}$$

for diagonal Δ_1 and Δ_2, and if d_0 minimizes the expression (3.4). For a design d, we have used N_{1d} to denote its treatment-row incidence matrix and N_{2d} its treatment-column incidence matrix. In the following corollary $(A)_{ij}$ denotes the (i,j)th entry of a matrix A.

Corollary 4.1. Suppose $r \equiv 0(\mathrm{mod}(u + \sqrt{uw}))$, $r \equiv 0(\mathrm{mod}(w+\sqrt{uw}))$, $c \equiv 0(\mathrm{mod}(u+\sqrt{uw})$ and $c \equiv 0(\mathrm{mod}(w+\sqrt{uw}))$. Let $d_0 \in \mathcal{D}(u,w,r,c)$ satisfy

$$(N_{1d_0})_{gj} = c/(u+\sqrt{uw}),\ (N_{1d_0})_{hj} = c/(w+\sqrt{uw}),$$
$$(N_{2d_0})_{gt} = r/(u+\sqrt{uw}),\ (N_{2d_0})_{ht} = r/(w+\sqrt{uw}), \tag{4.3}$$

for $g = 1,\ldots, u$, $h = u + 1,\ldots, v$, $j = 1,\ldots,r$, $t = 1,\ldots,c$. Then d_0 is A-optimal in $\mathcal{D}(u,w,r,c)$.

This corollary can be proved along the lines of Corollary 3.2. Let us describe one class of designs which satisfy (4.3). Let m be an integer and consider a latin square of order $s = m(w+\sqrt{uw})$. Change any m symbols of the latin square to A, any other m symbols to B, and so on. A,B,... denote the treatments in H, while 1,2,... denote the treatments in G. Change $m(\sqrt{w/u})$ symbols to 1, $m(\sqrt{w/u})$ symbols to 2, and so on. Call the resulting square L_{11}. Obtain L_{12} by starting with the same or a different square, and doing similar operations. Get squares L_{ij}, $i = 1,2,\ldots, q_1$, $j = 1,2,\ldots,q_2$, in this fashion. Form the $sq_1 \times sq_2$ array $L = ((L_{ij}))$. Then L is an A-optimal design in $\mathcal{D}(u,w, sq_1, sq_2)$. There are plenty of pairs (u,w) for which this method can be used - for example $u = 4$, $w = 16$.

This method of construction is a development on a method used by Notz (1984) for control-treatment comparisons. Indeed, Corollary 4.1 with $u = 1$ generalizes Corollary 2.1 of Notz. Our proof is also different. Corollary 2.1 of Notz states that if one starts with a latin square of order $p^2 + p$ and changes the symbols $p^2 + 1,\ldots,p^2 + p$ to 0 (the control) then the resulting design is A-optimal in $\mathcal{D}(1,p^2,p^2+p,p^2+p)$.

We can go a little further than this result of Notz for control-treatment comparisons. If $L_{ij}(i = 1,2,\ldots, q_1;$ $j = 1,2,\ldots,q_2)$ are $(p^2+p) \times (p^2+ p)$ squares obtained by Corollary 2.1 of Notz, then the array $((L_{ij}))$ is A-optimal in $\mathcal{D}(1,p^2,(p^2+p)q_1,(p^2+p)q_2)$. For example, let $p = 2$, $q_1 = 2$ and $q_2 = 3$. Then this method will give a patchwork of six 6×6 squares which is A-optimal in $\mathcal{D}(1,4,12,18)$. There are other methods of obtaining an A-optimal design too. For example, the layout,

<table>
<tr><td>Co</td><td>S_{11}</td><td>S_{12}</td><td rowspan="2">L_1</td></tr>
<tr><td>S_{21}</td><td>Co</td><td>S_{22}</td></tr>
<tr><td>S_{31}</td><td>S_{32}</td><td>Co</td><td>L_2</td></tr>
</table>

is A-optimal in $\mathcal{D}(1,4,12,18)$. Here Co is a 4×4 square consisting only of the control. S_{ij} $(i = 1,2,3\ ;\ j = 1,2,3)$ are 4×4 latin squares consisting of test treatments only. L_1 and L_2 are 6×6 squares obtained by Corollary 2.1 of Notz (1984).

The optimal designs of Corollaries 3.2 and 4.1 have a certain model robustness property. For instance, designs specified by (4.3) remain A-optimal, by Corollary 3.2, in case row effects are absent $(\rho_j = 0,\ j = 1, 2, \ldots, r)$, and even when column effects are also absent $(\rho_j = 0,\ j = 1,2,\ldots,r;\ \eta_t = 0,\ t = 1,2,\ldots,c)$.

The technique of finding optimal designs, given in Theorem 3.1, can be easily extended to additive models eliminating several sources of heterogeneity. This would generalize Corollary 4.1.

Finally we would like to point out that all the A-optimal designs given in this article are also optimal under several other criteria. One of them is MV-optimality. An MV optimal design minimizes

$$\max_{g \in G,\ h \in H} \operatorname{Var}(\hat{\tau}_{dg} - \hat{\tau}_{dh})$$

among all $d \in \mathcal{D}(u,w,b,k)$ or all $d \in \mathcal{D}(u,w,r,c)$, as the case may be.

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TABLE 1 Showing Optimal b_0, b_1 And b_2 When $k = 2$, For Various u, w AND $b \le 100$.

u	w	b	b_0	b_1	b_2
2	2	4	0	4	0
2	2	10	1	8	1
2	2	14	1	12	1
2	2	20	2	16	2
2	2	24	2	20	2
2	2	28	2	24	2
2	2	30	3	24	3
2	2	34	3	28	3
2	2	38	3	32	3
2	2	44	4	36	4
2	2	48	4	40	4
2	2	52	4	44	4
2	2	54	5	44	5
2	2	58	5	48	5
2	2	62	5	52	5
2	2	68	6	56	6
2	2	72	6	60	6
2	2	78	7	64	7
2	2	82	7	68	7
2	2	86	7	72	7
2	2	88	8	72	8
2	2	92	8	76	8
2	2	96	8	80	8

u	w	b	b_0	b_1	b_2
2	3	22	3	18	1
2	3	38	6	30	2
2	3	60	9	48	3
2	3	76	12	60	4
2	3	99	15	78	6
2	4	31	6	24	1
2	4	86	18	64	4
2	5	41	10	30	1
2	6	52	15	36	1
3	3	24	3	18	3
3	3	57	6	45	6
3	3	81	9	63	9
3	4	78	12	60	6

END

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